J Math Chem (2009) 46:1394–1408
DOI 10.1007/s10910-009-9523-7

ORIGINAL PAPER

Bifurcation analysis on a reactor model with combination of quadratic and cubic steps

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Received: 16 November 2008 / Accepted: 14 January 2009 / Published online: 31 January 2009
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Abstract In this paper we give a rigorous bifurcation analysis for a reactor model with combination of quadratic and cubic steps. We extend the analysis of Satnoianu et al. in two directions. First, we impose Dirichlet boundary conditions instead of the semi-infinite domain in one space dimension. Second, we consider a variety of different bifurcation phenomena of the corresponding steady state system, focusing on their parametric sensitivity. We show that the system exhibits subcritical pitchfork bifurcation, supercritical pitchfork bifurcation and transcritical bifurcation in the different regimes of control parameters. To compare the decades old work by Auchmuty et al., we show that the spatial structures formed by the parameter $p \in [0, 1]$ are richer than those formed by the purely quadratic step for $p = 0$ and the purely cubic step for $p = 1$ respectively.

Keywords Bifurcation · Reaction-diffusion · Subcritical pitchfork bifurcation · Supercritical pitchfork bifurcation · Transcritical bifurcation

1 Introduction

Reaction-diffusion models exhibit a great variety of spatial patterns. However, their relevance to biological systems has been criticized due to their extreme sensitivity to parameter values. Bifurcation theory enables one to determine how qualitatively different regimes appear and disappear as control parameters vary. For example, this kind of problems are discussed in [2,3,8,15–17], etc. Murray [8] showed that the parameter

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regime, in which a number of reaction-diffusion models gave rise to diffusion-driven instability, is really very small.

Satnoianu et al. [9–11] considered a model in which is assumed that a precursor P^+ is present in excess. To maintain electroneutrality they also required a further species Q^- to be present in the reactor at a concentration similar to that P^+ though this species does not take part in the reaction. Assume that P^+ decays at a constant rate to form the substrate A^+ via $P^+ \to A^+$, the rate is $k_0 P_0$, where P_0 is the initial concentration of the reservoir species P^+. The substrate A^+ and autocatalyst B^+ subsequently react by following a linear combination of quadratic and cubic according to the scheme

$$A^+ + B^+ \to 2B^+, \quad rate = k_q ab,$$
$$A^+ + 2B^+ \to 3B^+, \quad rate = k_c ab^2,$$
$$B^+ \to C^+, \quad rate = k_1 b,$$

where k_q, k_c, k_1 are constants and a, b are the concentrations of A^+ and B^+, respectively. This leads to the dimensionless reactor model

$$a_t = \delta a_{xx} - \delta \phi a_x + \mu - pab - (1-p)ab^2,$$
$$b_t = b_{xx} - \phi b_x + pab + (1-p)ab^2 - b$$

where a, b, t and x are the dimensionless concentrations, time and distance along the reactor, respectively. μ depends on the rates k_q, k_c, k_1. Here $\delta = D_A/D_B$ is the ratio of diffusion coefficients of substrate A^+ and autocatalyst B^+, and ϕ is a parameter denoting the intensity of applied electric field (this can also be interpreted as the dimensionless drift velocity, see Satnoianu et al. [11]). In the case $\phi \neq 0$, the difficulty is in estimating the spectrum of the linearized operator around the trivial solution, a rigorous study will be reported elsewhere. In this paper we just consider the problem for $\phi = 0$, that is,

$$\begin{aligned} a_t &= \delta a_{xx} + \mu - pab - (1-p)ab^2, \\ b_t &= b_{xx} + pab + (1-p)ab^2 - b, \end{aligned} \tag{1.1}$$

where the parameter $p \in [0, 1]$ measures the strength of the quadratic step against the cubic step. For $p = 1$, the nonlinearity in the kinetics is purely quadratic while for $p = 0$, it is purely cubic, the Selkov scheme for glycolysis [14], has also been used by Hunding [5] and Hunding and Sorensen [6] as a model in morphogenesis. In the work of [7], it extends solely to the bifurcation analysis for cubic step.

In this paper we extend the analysis of Satnoianu et al. [11] in two directions. First, we impose Dirichlet boundary conditions instead of the semi-infinite domain in one space dimension. Periodic solutions to the system (1.1) bifurcate from the steady state through a Hopf bifurcation. This leads us to study the stability of the steady state of the system (1.1) in terms of the spatial co-ordinate. Second, we consider a variety of different bifurcation phenomena of the corresponding steady state system, focusing on their parametric sensitivity. To compare the decades old work by Auchmuty et al. [1], we show that the spatial structures formed by the parameter $p \in [0, 1]$ are richer than those formed by the purely quadratic step for $p = 0$ and the purely cubic step for $p = 1$ respectively. In Sect. 2, we simplify the model and rewrite the system into a dimensionless one. In Sect. 3, we discuss the stability of the trivial solution and give the condition of the occurrence of bifurcation from the trivial solution. In Sect. 4, we consider the bifurcation directions. In Sect. 5, we give a discussion.

2 Simplification of the model

The system (1.1) has a spatially uniform steady state $(a, b) = \left(\frac{1}{p+(1-p)\mu}, \mu\right)$, which exists for $p \in [0, 1]$. Let

$$A = [p + (1-p)\mu]a, \quad B = \frac{b}{\mu}, \quad \tilde{t} = \mu[p + (1-p)\mu]t,$$
$$\tilde{x} = \sqrt{\mu[p + (1-p)\mu]}x,$$

then we have

$$\begin{aligned} A_{\tilde{t}} &= \delta A_{\tilde{x}\tilde{x}} + 1 - \frac{p}{p + (1-p)\mu} AB - \frac{(1-p)\mu}{p + (1-p)\mu} AB^2, \\ B_{\tilde{t}} &= B_{\tilde{x}\tilde{x}} + \frac{1}{\mu[p + (1-p)\mu]} \left[\frac{p}{p + (1-p)\mu} AB + \frac{(1-p)\mu}{p + (1-p)\mu} AB^2 - B \right]. \end{aligned} \tag{2.1}$$

Denote $q = \frac{p}{p+(1-p)\mu}$ and $c = \frac{1}{\mu[p+(1-p)\mu]}$, the system (2.1) becomes, on dropping the tildes for notational convenience,

$$\begin{aligned} A_t &= \delta A_{xx} + 1 - qAB - (1-q)AB^2, \\ B_t &= B_{xx} + c[qAB + (1-q)AB^2 - B]. \end{aligned} \tag{2.2}$$

The steady state solutions of (2.2) satisfy that

$$\begin{aligned} \delta A_{xx} + 1 - qAB - (1-q)AB^2 &= 0, \\ B_{xx} \quad + c[qAB + (1-q)AB^2 - B] &= 0. \end{aligned} \tag{2.3}$$

Our aim is to study the bifurcation solutions of the elliptic system

$$\begin{aligned} &\delta A''(x) + 1 - qAB - (1-q)AB^2 = 0, \quad x \in (0, L), \\ &B''(x) + c[qAB + (1-q)AB^2 - B] = 0, \\ &A(0) = A(L) = B(0) = B(L) = 1, \end{aligned} \tag{2.4}$$

where the boundary conditions match the constant solution $A(x) = B(x) = 1$, $L > 0$ is a constant. Let $u = A - 1$, $v = B - 1$, then

$$\begin{aligned} \delta u''(x) + 1 - q(u+1)(v+1) - (1-q)(u+1)(v+1)^2 = 0, \quad x \in (0, L), \\ v''(x) + c[q(u+1)(v+1) + (1-q)(u+1)(v+1)^2 - v - 1] = 0. \end{aligned}$$

Denote $d = \frac{1}{c\delta}$, $\lambda = \frac{L^2}{\delta}$, we have

$$\begin{aligned} &u''(x) + \lambda[1 - q(u+1)(v+1) - (1-q)(u+1)(v+1)^2] = 0, \quad x \in (0, 1), \\ &dv''(x) + \lambda[q(u+1)(v+1) + (1-q)(u+1)(v+1)^2 - v - 1] = 0, \\ &u(0) = u(1) = v(0) = v(1) = 0. \end{aligned} \tag{2.5}$$

Introduce a linear operator

$$\mathcal{L}(\lambda, d) = \begin{pmatrix} \frac{d^2}{dx^2} - \lambda & (q-2)\lambda \\ \lambda & d\frac{d^2}{dx^2} + \lambda(1-q) \end{pmatrix},$$

then the system (2.5) becomes

$$\begin{aligned} &\mathcal{L}(\lambda, d)\begin{pmatrix} u \\ v \end{pmatrix} + \lambda N(u, v)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x \in (0, 1) \\ &u(0) = u(1) = v(0) = v(1) = 0, \end{aligned} \tag{2.6}$$

where $N(u, v) = N^{(2)}(u, v) + N^{(3)}(u, v)$ and

$$N^{(2)}(u, v) = (2-q)uv + (1-q)v^2, \quad N^{(3)}(u, v) = (1-q)uv^2.$$

Consider the Banach spaces $U = C_0^2[0, 1]$ and $V = C[0, 1]$, we have

$$\mathcal{L}(\lambda, d)\begin{pmatrix} u \\ v \end{pmatrix} + \lambda \mathcal{N}(u, v)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in U^2, \tag{2.7}$$

where $\mathcal{N}$ is the corresponding nonlinear operator from N.

3 Bifurcation from the trivial solution

It is obvious that $(0, 0)^T$ is a trivial solution of (2.7) for the parameters $(\lambda, d) \in (0, +\infty) \times (0, +\infty)$ and $q \in [0, 1]$. The linearized system of (2.7) at the trivial solution $(0, 0)^T$ is

$$\mathcal{L}(\lambda, d)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in U^2. \tag{3.1}$$

Denote $\mu_n = n^2\pi^2$ and $\varphi_n(x) = sin(n\pi x)$, $x \in (0, 1)$, $n = 1, 2, ...,$ let

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \varphi_n(x), \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in R^2,$$

then we have

$$\begin{pmatrix} \frac{d^2}{dx^2} - \lambda & (q-2)\lambda \\ \lambda & d\frac{d^2}{dx^2} + \lambda(1-q) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \varphi_n(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is,

$$\begin{pmatrix} -\mu_n - \lambda & (q-2)\lambda \\ \lambda & -d\mu_n + \lambda(1-q) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

(3.2) has non-trivial solution $(\alpha, \beta) \in R^2$ if and only if $(\lambda, d) \in H_n$, $n \geq 1$, where

$$H_n = \{(\lambda, d) \in (0, +\infty)^2 : (\lambda + \mu_n)(d\mu_n - \lambda(1-q)) + (2-q)\lambda^2 = 0\}.$$

In fact, if $(\lambda, d) \in H_n$, we have

$$\Phi_n(\lambda, d) = \begin{pmatrix} \frac{(q-2)\lambda}{\mu_n+\lambda} \\ 1 \end{pmatrix} \varphi_n(x) \in N[\mathcal{L}(\lambda, d)]. \tag{3.3}$$

For $n \geq 1$, denote

$$d_n(\lambda) = \frac{\mu_n(1-q)\lambda - \lambda^2}{\mu_n(\lambda + \mu_n)}, \quad (\lambda, d) \in H_n. \tag{3.4}$$

If $0 < \lambda < \mu_n(1-q)$, $d_n(\lambda)$ attains the maximum at $\lambda = (\sqrt{2-q} - 1)\mu_n$, and its maximum $max d_n(\lambda) = (\sqrt{2-q} - 1)^2$ does not depend on n.

Lemma 3.1 *Given $(\lambda_0, d_0) \in (0, +\infty)^2$, the space $N[\mathcal{L}(\lambda_0, d_0)]$ is non-trivial if and only if $(\lambda_0, d_0) \in H_n$ for some integer $n \geq 1$. In that case, the following two situations can occur.*

1. $(\lambda_0, d_0) \in H_n \setminus \cup_{k=1,k\neq n}^{\infty} H_k$, *and then*

$$N[\mathcal{L}(\lambda_0, d_0)] = Span\{\Phi_n(\lambda_0, d_0)\}, \tag{3.5}$$

$$N[\mathcal{L}^*(\lambda_0, d_0)] = Span\{\Phi_n^*(\lambda_0, d_0)\}, \tag{3.6}$$

$$R[\mathcal{L}(\lambda_0, d_0)] = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in V^2 : \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \Phi_n^*(\lambda_0, d_0) \right\rangle = 0 \right\} = N[\mathcal{L}^*(\lambda_0, d_0)]^{\perp},$$

where $\mathcal{L}^*(\lambda_0, d_0)$ is the adjoint operator of $\mathcal{L}(\lambda_0, d_0)$, and

$$\Phi_n^*(\lambda, d) = \begin{pmatrix} \frac{\lambda}{\mu_n+\lambda} \\ 1 \end{pmatrix} \varphi_n(x), \quad \Phi_n^*(\lambda_0, d_0) \in N[\mathcal{L}^*(\lambda_0, d_0)], \tag{3.7}$$

the angular brackets stand for the L^2-*duality*

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle = \int_0^1 [u_1(x)u_2(x) + v_1(x)v_2(x)]dx. \tag{3.8}$$

2. $(\lambda_0, d_0) \in H_n \cap H_m, m \neq n$, *and then*

$$N[\mathcal{L}(\lambda_0, d_0)] = Span\{\Phi_n(\lambda_0, d_0), \Phi_m(\lambda_0, d_0)\}, \tag{3.9}$$

$$N[\mathcal{L}^*(\lambda_0, d_0)] = Span\{\Phi_n^*(\lambda_0, d_0), \Phi_m^*(\lambda_0, d_0)\}, \tag{3.10}$$

$$R[\mathcal{L}(\lambda_0, d_0)] = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in V^2 : 0 = \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \Phi_n^*(\lambda_0, d_0) \right\rangle = \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \Phi_m^*(\lambda_0, d_0) \right\rangle \right\}.$$

Remark (i) $\forall n \geq 1, m \geq 1, m \neq n, H_n \cap H_m = \{(\lambda_{nm}, d_{nm})\}$, where

$$\lambda_{nm} = \frac{1}{2}\left[-(\mu_n + \mu_m) \pm \sqrt{(\mu_n + \mu_m)^2 + 4\mu_n\mu_m(1-q)}\right], \ d_{nm} = d_n(\lambda_{nm}).$$

(ii) $\forall n > m > r, H_n \cap H_m \cap H_r = \phi$.
In the remainder of this section we fix

$$d_0 \in \left(0, (\sqrt{2-q} - 1)^2\right), \tag{3.11}$$

consider λ_0 such that for some integer $n \geq 1$, $(\lambda_0, d_0) \in H_n \backslash \cup_{k=1, k\neq n}^{\infty} H_k$. Regarding λ as bifurcation parameter, we prove that $(\lambda, u, v) = (\lambda_0, 0, 0)$ is a bifurcation point of (2.7).

Here we give some definitions and results described in [3,7].

Definition 3.1 (*Definition 2.1.3 in [7]*) Let $A, B \in \mathcal{L}(U, V)$. The resolvent set $\rho(A, B)$ of the pair (A, B) is defined as the set of $\lambda \in C$ for which $A - \lambda B$ has a bounded inverse. The spectrum $\sigma(A, B)$ of (A, B) is defined as $\sigma(A, B) = C \backslash \rho(A, B)$. $\lambda_0 \in \sigma(A, B)$ is said to be an eigenvalue of (A, B) if zero is an eigenvalue of $A - \lambda_0 B$. $\lambda_0 \in C$ is said to be a simple eigenvalue of (A, B) if $dim N[A - \lambda_0 B] = codim R[A - \lambda_0 B] = 1$ and $B(N[A - \lambda_0 B]) \bigoplus R[A - \lambda_0 B] = V$.

Rewrite the system (2.7) as the follows

$$\mathcal{L}_0(\lambda, d)\begin{pmatrix} u \\ v \end{pmatrix} + (\lambda - \lambda_0)\mathcal{L}_1(\lambda, d)\begin{pmatrix} u \\ v \end{pmatrix} + \lambda \mathcal{N}(u, v)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in U^2, \tag{3.12}$$

where

$$\mathcal{L}_0 = \mathcal{L}(\lambda_0, d_0), \quad \mathcal{L}_1 = \begin{pmatrix} -1 & q-2 \\ 1 & 1-q \end{pmatrix}.$$

Lemma 3.2 *(Theorem 2.2.1 in [7]) Suppose $F(\lambda, u)$ is of class C^r for some $r \geq 2$ and zero is a simple eigenvalue of the pair $(\mathcal{L}_0, \mathcal{L}_1)$. Let $Y \subset U$ be a subspace such that $N(\mathcal{L}_0) \bigoplus Y = U$. Then, there exist $\varepsilon > 0$ and maps of class C^{r-1}*

$$\lambda : (-\varepsilon, \varepsilon) \to R, \quad y : (-\varepsilon, \varepsilon) \to Y$$

such that $\lambda(0) = \lambda_0$, $y(0) = 0$ and for each $s \in (-\varepsilon, \varepsilon)$,

$$F(\lambda(s), u(s)) = 0, \quad u(s) = s(\varphi_0 + y(s)).$$

Following the Lemma 3.2, we obtain the existence result for the system (3.12).

Theorem 3.1 *There exist $\varepsilon > 0$ and an analytic map $(\lambda, u, v) : (-\varepsilon, \varepsilon) \to R \times U^2$ such that $(\lambda(0), u(0), v(0)) = (\lambda_0, 0, 0)$ and $(\lambda(s), u(s), v(s))$ is a solution of (3.12) for each $s \in (-\varepsilon, \varepsilon)$. Moreover, this curve can be chosen so that if we write*

$$\lambda(s) = \sum_{j=0}^{\infty} \lambda_j s^j, \quad \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = s \left[\Phi_n(\lambda_0, d_0) + \sum_{j=1}^{\infty} s^j \begin{pmatrix} u_j \\ v_j \end{pmatrix} \right], \tag{3.13}$$

then

$$\left\langle \begin{pmatrix} u_j \\ v_j \end{pmatrix}, \Phi_n(\lambda_0, d_0) \right\rangle = 0, \quad \forall j \geq 1. \tag{3.14}$$

Furthermore, any non-trivial solution of (3.12) in some open neighborhood of $(\lambda_0, 0, 0)$ must be of the form $(\lambda(s), u(s), v(s))$ for some $s \simeq 0$ and $s \neq 0$.

Proof By direct computation, we have

$$\begin{aligned} \langle \mathcal{L}_1 \Phi_n(\lambda_0, d_0), \Phi_n^*(\lambda_0, d_0) \rangle &= \left\langle \begin{pmatrix} \frac{(2-q)\lambda_0}{\mu_n+\lambda_0} + q - 2 \\ \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} + 1 - q \end{pmatrix} \varphi_n(x), \begin{pmatrix} \frac{\lambda_0}{\mu_n+\lambda_0} \\ 1 \end{pmatrix} \varphi_n(x) \right\rangle \\ &= \int_0^1 \left[\frac{\mu_n(q-2)\lambda_0}{(\mu_n+\lambda_0)^2} + \frac{\mu_n(1-q)-\lambda_0}{\mu_n+\lambda_0} \right] \varphi_n^2(x)dx \\ &= \frac{-1}{2(\mu_n+\lambda_0)^2} \left[\lambda_0 + (1-\sqrt{2-q})\mu_n \right] \left[\lambda_0 + (1+\sqrt{2-q})\mu_n \right], \end{aligned}$$

and $d_0 < (\sqrt{2-q} - 1)^2$ implies that $\lambda_0 \neq (\sqrt{2-q} - 1)\mu_n$, then

$$\langle \mathcal{L}_1 \Phi_n(\lambda_0, d_0), \Phi_n^*(\lambda_0, d_0) \rangle \neq 0,$$

it means that $\mathcal{L}_1 \Phi_n(\lambda_0, d_0)$ does not belong to $R(\mathcal{L}_0)$, so we have

$$\mathcal{L}_1(N(\mathcal{L}_0)) \bigoplus R(\mathcal{L}_0) = V^2. \tag{3.15}$$

Due to the definition 3.1, we know that zero is a simple eigenvalue of the pair $(\mathcal{L}_0, \mathcal{L}_1)$ and (3.13) implies that the transversality condition is satisfied (Proposition 2.1.2 in [7]), then the results in the theorem come from the Lemma 3.2. This completes the proof. □

4 Bifurcation direction

In this section we give the bifurcation directions. Since

$$\begin{aligned}
N^{(2)}(u,v) &= (2-q)uv + (1-q)v^2 \\
&= (2-q)\left[s\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}\varphi_n + u_1 s^2 + o(s^2)\right]\left[s\varphi_n + v_1 s^2 + o(s^2)\right] \\
&\quad + (1-q)\left[s\varphi_n + v_1 s^2 + o(s^2)\right]^2 \\
&= s^2 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right)\varphi_n^2 + s^3 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) v_1\varphi_n \\
&\quad + s^3\left[(2-q)u_1 + (1-q)v_1\right]\varphi_n + O(s^4), \\
N^{(3)}(u,v) &= (1-q)uv^2 \\
&= (1-q)\left[s\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}\varphi_n + u_1 s^2 + o(s^2)\right]\left[s\varphi_n + v_1 s^2 + o(s^2)\right]^2 \\
&= (1-q)\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}\varphi_n^3 s^3 + O(s^4) \\
&= s^3 N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right)\varphi_n^3 + O(s^4).
\end{aligned}$$

From (3.12) we have

$$\begin{aligned}
&\mathcal{L}_0\left[s\Phi_n + s^2\begin{pmatrix}u_1\\v_1\end{pmatrix} + s^3\begin{pmatrix}u_2\\v_2\end{pmatrix} + O(s^4)\right] \\
&+\left[\lambda_1 s + \lambda_2 s^2 + O(s^3)\right]\mathcal{L}_1\left[s\Phi_n + s^2\begin{pmatrix}u_1\\v_1\end{pmatrix} + s^3\begin{pmatrix}u_2\\v_2\end{pmatrix} + O(s^4)\right] \\
&+\left[\lambda_0 + \lambda_1 s + \lambda_2 s^2 + O(s^3)\right]\left[N^{(2)}(u,v) + N^{(3)}(u,v)\right]\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},
\end{aligned}$$

(3.5) implies that $\mathcal{L}_0\Phi_n = 0$, then we have

$$\mathcal{L}_0\begin{pmatrix}u_1\\v_1\end{pmatrix} + \lambda_1\mathcal{L}_1\Phi_n + \lambda_0 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right)\varphi_n^2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}; \tag{4.1}$$

$$\begin{aligned}
&\mathcal{L}_0\begin{pmatrix}u_2\\v_2\end{pmatrix} + \lambda_1\mathcal{L}_1\begin{pmatrix}u_1\\v_1\end{pmatrix} + \lambda_2\mathcal{L}_1\Phi_n + \lambda_0 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) v_1\varphi_n\begin{pmatrix}-1\\1\end{pmatrix} \\
&+\lambda_0[(2-q)u_1 + (1-q)v_1]\varphi_n\begin{pmatrix}-1\\1\end{pmatrix} + \lambda_0 N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right)\varphi_n^3\begin{pmatrix}-1\\1\end{pmatrix}
\end{aligned}$$

$$+\lambda_1 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0},1\right)\varphi_n^2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}; \tag{4.2}$$

$$\ldots$$

Taking the scalar product of (4.1) with $\Phi_n^*(\lambda_0, d_0)$ provides

$$\left\langle \mathcal{L}_0\begin{pmatrix}u_1\\v_1\end{pmatrix}, \Phi_n^*\right\rangle + \lambda_1\left\langle \mathcal{L}_1\Phi_n, \Phi_n^*\right\rangle + \lambda_0 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0},1\right)$$
$$\left\langle \varphi_n^2\begin{pmatrix}-1\\1\end{pmatrix}, \Phi_n^*\right\rangle = 0.$$

Since

$$\left\langle \mathcal{L}_0\begin{pmatrix}u_1\\v_1\end{pmatrix}, \Phi_n^*(\lambda_0,d_0)\right\rangle = \left\langle \begin{pmatrix}u_1\\v_1\end{pmatrix}, \mathcal{L}_0^*\Phi_n^*(\lambda_0,d_0)\right\rangle = 0;$$
$$\left\langle \mathcal{L}_1\Phi_n(\lambda_0,d_0), \Phi_n^*(\lambda_0,d_0)\right\rangle = \int_0^1\left[\frac{(q-2)\lambda_0\mu_n}{(\mu_n+\lambda_0)^2}+\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}+1-q\right]\varphi_n^2 dx$$
$$=\frac{-1}{2(\mu_n+\lambda_0)^2}\left[\lambda_0+(1-\sqrt{2-q})\mu_n\right]\left[\lambda_0+(1+\sqrt{2-q})\mu_n\right];$$
$$\left\langle \varphi_n^2\begin{pmatrix}-1\\1\end{pmatrix}, \Phi_n^*(\lambda_0,d_0)\right\rangle = \frac{\mu_n}{\mu_n+\lambda_0}\int_0^1\varphi_n^3 dx,$$

we have

$$\lambda_1(\lambda_0) = \frac{-\lambda_0}{<\mathcal{L}_1\Phi_n, \Phi_n^*(\lambda_0,d_0)>}N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0},1\right)\left\langle \varphi_n^2\begin{pmatrix}-1\\1\end{pmatrix}, \Phi_n^*(\lambda_0,d_0)\right\rangle$$
$$=\frac{2\mu_n\lambda_0(\mu_n+\lambda_0)}{\left[\lambda_0+(1-\sqrt{2-q})\mu_n\right]\left[\lambda_0+(1+\sqrt{2-q})\mu_n\right]}N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0},1\right)\int_0^1\varphi_n^3 dx.$$

As

$$\int_0^1\varphi_n^3 dx = \int_0^1 sin^3(n\pi x)dx = \frac{2}{3n\pi}[1-(-1)^n], \quad n\geq 1,$$

we must distinguish two different situations according to the value of n.

Case 1 Suppose n is an odd natural number, then we have

$$\lambda_1(\lambda_0) = \frac{-8\lambda_0\mu_n\left[(q^2-3q+3)\lambda_0-(1-q)\mu_n\right]}{3n\pi[\lambda_0+(1-\sqrt{2-q})\mu_n]\left[\lambda_0+(1+\sqrt{2-q})\mu_n\right]}. \tag{4.3}$$

Denote $q_0\in(0,1)$ the unique root of equation $(1-q)\sqrt{2-q}=1$ on $[0,1]$. In fact, the function $f(q)=(1-q)\sqrt{2-q}-1$ is continuous on $[0,1]$ and $f(1)=-1<0$, $f(0)=\sqrt{2}-1>0$, $f'(q)=(3q-5)/(2\sqrt{2-q})<0$ for $q\in(0,1)$.

If $0 < q < q_0$, we have

$$\begin{aligned} q^2 - 3q + 3 - \left(\sqrt{2-q}+1\right) &= \sqrt{2-q}\left[(1-q)\sqrt{2-q}-1\right] \\ &> \sqrt{2-q}\left[(1-q_0)\sqrt{2-q_0}-1\right] = 0, \end{aligned}$$

that is, $q^2 - 3q + 3 > \sqrt{2-q} + 1$, hence

$$\frac{1-q}{q^2-3q+3}\mu_n < \frac{1-q}{\sqrt{2-q}+1}\mu_n = \left(\sqrt{2-q}-1\right)\mu_n.$$

Therefore, we get the following subcases.

(1) $0 < \lambda_0 < \frac{1-q}{q^2-3q+3}\mu_n$ implies $\lambda_1(\lambda_0) < 0$. In fact, $(q^2-3q+3)\lambda_0-(1-q)\mu_n < 0$, $\lambda_0 < \left(\sqrt{2-q}-1\right)\mu_n$, from (4.3) we get $\lambda_1(\lambda_0) < 0$.
(2) $\lambda_0 = \frac{1-q}{q^2-3q+3}\mu_n$ implies $\lambda_1(\lambda_0) = 0$.
(3) $\frac{1-q}{q^2-3q+3}\mu_n < \lambda_0 < \left(\sqrt{2-q}-1\right)\mu_n$ implies $\lambda_1(\lambda_0) > 0$.
(4) $\left(\sqrt{2-q}-1\right)\mu_n < \lambda_0 < (1-q)\mu_n$ implies $\lambda_1(\lambda_0) < 0$.

To sum up, the bifurcation direction is $\lambda = \lambda_0 + \lambda_1 s + O(s^2)$ if $\lambda_0 \neq \frac{1-q}{q^2-3q+3}\mu_n$, it implies that every bifurcation is transcritical.

If $\lambda_0 = \frac{1-q}{q^2-3q+3}\mu_n$, then $\lambda_1(\lambda_0) = 0$, and

$$N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) = \frac{1}{\mu_n+\lambda_0}\left[(1-q)\mu_n - (q^2-3q+3)\lambda_0\right] = 0,$$

system (4.1) becomes $\mathcal{L}_0\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0$, hence $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in N(\mathcal{L}_0)$. As $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in N(\mathcal{L}_0)^\perp$, we have $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Substitute this into the system (4.2), we have

$$\mathcal{L}_0\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \lambda_2\mathcal{L}_1\Phi_n + \lambda_0 N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right)\varphi_n^3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (4.4)$$

Taking the scalar product of (4.4) with $\Phi_n^*(\lambda_0, d_0)$ provides

$$\begin{aligned} &\left\langle \mathcal{L}_0\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \Phi_n^* \right\rangle + \lambda_2\left\langle \mathcal{L}_1\Phi_n, \Phi_n^*\right\rangle + \lambda_0 N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) \\ &\left\langle \varphi_n^3\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \Phi_n^* \right\rangle = 0. \end{aligned}$$

Since $\left\langle \mathcal{L}_0 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \Phi_n^* \right\rangle = \left\langle \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \mathcal{L}_0^* \Phi_n^* \right\rangle = 0$, and

$$\int_0^1 \varphi_n^4 dx = \int_0^1 sin^4(n\pi x)dx = \frac{3}{8}, \quad N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) = (1-q)\frac{(q-2)\lambda_0}{\lambda_0+\mu_n},$$

we have

$$\begin{aligned}
\lambda_2(\lambda_0) &= \frac{-\lambda_0}{\langle \mathcal{L}_1 \Phi_n, \Phi_n^* \rangle} N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) \left\langle \varphi_n^3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \Phi_n^*(\lambda_0, d_0) \right\rangle \\
&= \frac{-\lambda_0}{\langle \mathcal{L}_1 \Phi_n, \Phi_n^* \rangle} N^{(3)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) \left(1 - \frac{\lambda_0}{\lambda_0+\mu_n}\right) \int_0^1 \varphi_n^4 dx \\
&= \frac{3(1-q)(q-2)\lambda_0^2 \mu_n}{4[\lambda_0 + (1-\sqrt{2-q})\mu_n][\lambda_0 + (1+\sqrt{2-q})\mu_n]}.
\end{aligned} \tag{4.5}$$

In this case, $\lambda_0 = \frac{1-q}{q^2-3q+3}\mu_n < (\sqrt{2-q}-1)\mu_n$, then $\lambda_2(\lambda_0) > 0$, as the bifurcation direction is $\lambda = \lambda_0 + \lambda_2 s^2 + o(s^2)$, therefore, at (λ_0, d_0), the local bifurcation diagram is a supercritical pitchfork.

If $q_0 < q < 1$, we have

$$\begin{aligned}
q^2 - 3q + 3 - (\sqrt{2-q} + 1) &= \sqrt{2-q}\left[(1-q)\sqrt{2-q} - 1\right] \\
&< \sqrt{2-q}\left[(1-q_0)\sqrt{2-q_0} - 1\right] = 0,
\end{aligned}$$

that is, $q^2 - 3q + 3 < \sqrt{2-q} + 1$, therefore, from (4.3) we get the following subcases.

(1) $0 < \lambda_0 < (\sqrt{2-q}-1)\mu_n$ implies $\lambda_1(\lambda_0) < 0$.
(2) $(\sqrt{2-q}-1)\mu_n < \lambda_0 < \frac{1-q}{q^2-3q+3}\mu_n$ implies $\lambda_1(\lambda_0) > 0$.
(3) $\lambda_0 = \frac{1-q}{q^2-3q+3}\mu_n$ implies $\lambda_1(\lambda_0) = 0$.
(4) $\frac{1-q}{q^2-3q+3}\mu_n < \lambda_0 < (1-q)\mu_n$ implies $\lambda_1(\lambda_0) < 0$.

To sum up, the bifurcation direction is $\lambda = \lambda_0 + \lambda_1 s + O(s^2)$ if $\lambda_0 \neq \frac{1-q}{q^2-3q+3}\mu_n$, every bifurcation is transcritical.

If $\lambda_0 = \frac{1-q}{q^2-3q+3}\mu_n$, then $\lambda_1(\lambda_0) = 0$, but in this case, $\lambda_0 = \frac{1-q}{q^2-3q+3}\mu_n > (\sqrt{2-q}-1)\mu_n$, (4.5) implies $\lambda_2(\lambda_0) < 0$, as the bifurcation direction is $\lambda = \lambda_0 + \lambda_2 s^2 + o(s^2)$, therefore, at (λ_0, d_0), the local bifurcation diagram is a subcritical pitchfork.

Case 2 Suppose n is an even natural number, then we have
$\int_0^1 \varphi_n^3 dx = \int_0^1 sin^3(n\pi x)dx = 0$, therefore, $\lambda_1(\lambda_0) = 0$ and the system (4.1) is reduced into

$$\mathcal{L}_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \lambda_0 N^{(2)}\left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1\right) \varphi_n^2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.6}$$

subject to the conditions

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in U^2, \quad \left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \Phi_n(\lambda_0, d_0) \right\rangle = 0. \tag{4.7}$$

Since

$$\left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \varphi_n^2, \Phi_n(\lambda_0, d_0) \right\rangle = \frac{\mu_n}{\mu_n + \lambda_0} \int_0^1 \varphi_n^3 dx = 0,$$

by the Fredholm alternative theorem, (4.6) subject to the constraint of (4.7) has a unique solution. Now we find this solution. Since

$$\begin{aligned} \mathcal{L}_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} &= -\lambda_0 N^{(2)} \left(\frac{(q-2)\lambda_0}{\mu_n + \lambda_0}, 1 \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \varphi_n^2(x) \\ &= \frac{\lambda_0}{2} N^{(2)} \left(\frac{(q-2)\lambda_0}{\mu_n + \lambda_0}, 1 \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} [1 - cos(2n\pi x)], \end{aligned}$$

that is,

$$\begin{aligned} \mathcal{L}_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} &= \begin{pmatrix} \frac{d^2}{dx^2} - \lambda_0 & (q-2)\lambda_0 \\ \lambda_0 & d_0 \frac{d^2}{dx^2} + \lambda_0(1-q) \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\ &= \begin{pmatrix} A \\ -A \end{pmatrix} - \begin{pmatrix} A \\ -A \end{pmatrix} cos(2n\pi x), \end{aligned} \tag{4.8}$$

where $A = \frac{\lambda_0}{2} N^{(2)} \left(\frac{(q-2)\lambda_0}{\mu_n+\lambda_0}, 1 \right)$. By the superposition principle and the method of undetermined coefficients, (4.8) has a particular solution

$$\begin{pmatrix} u_{1,p} \\ v_{1,p} \end{pmatrix} = \begin{pmatrix} \frac{-A}{\lambda_0} \\ 0 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} A(\lambda_0 + \mu_2 n d_0) \\ -A\mu_2 n \end{pmatrix} cos(2n\pi x),$$

where $k = -(\lambda_0 + \mu_{2n})[\lambda_0(1-q) - \mu_{2n} d_0] - (q-2)\lambda_0^2$, due to the condition $(\lambda_0, d_0) \in H_n \backslash H_{2n}$, we know that $k \neq 0$. It is easy to verify that $\left\langle \begin{pmatrix} u_{1,p} \\ v_{1,p} \end{pmatrix}, \Phi_n(\lambda_0, d_0) \right\rangle = 0$.

Now we consider the homogeneous system

$$\begin{pmatrix} \frac{d^2}{dx^2} - \lambda_0 & (q-2)\lambda_0 \\ \lambda_0 & d_0 \frac{d^2}{dx^2} + \lambda_0(1-q) \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.9}$$

subject to the constraint condition

$$\left\langle \begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix}, \Phi_n(\lambda_0, d_0) \right\rangle = 0. \tag{4.10}$$

The general solution of (4.9) is

$$\begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} = [C_1 cos(n\pi x) + C_2 sin(n\pi x)]A \begin{pmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} \\ 1 \end{pmatrix} + [C_3 cos(\Omega x) + C_4 sin(\Omega x)]A \begin{pmatrix} \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} \\ 1 \end{pmatrix}, \tag{4.11}$$

where $\Omega = \sqrt{\frac{\lambda_0(\mu_n+\lambda_0)}{\mu_n(1-q)-\lambda_0}}$, for $0 < d_0 < (\sqrt{2-q} - 1)^2$, we know that $0 < \lambda_0 < (1-q)\mu_n$ and $\lambda_0 \neq (\sqrt{2-q} - 1)\mu_n$, thus $\Omega \neq n\pi = \sqrt{\mu_n}$. By the condition (4.10), we have

$$\frac{1}{2}\left[\frac{(q-2)^2\lambda_0^2}{(\mu_n+\lambda_0)^2} + 1\right]C_2 + \frac{n\pi}{n^2\pi^2 - \Omega^2}\left[\frac{(q-2)^2\lambda_0^2}{(\mu_n+\lambda_0)(\Omega^2+\lambda_0)} + 1\right][(1 - cos\Omega)C_3 - sin\Omega C_4] = 0. \tag{4.12}$$

Thus, the solution which satisfies the condition (4.10) of the system (4.8) is

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = A[C_1 cos(n\pi x) + C_2 sin(n\pi x)] \begin{pmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} \\ 1 \end{pmatrix} + A[C_3 cos(\Omega x) + C_4 sin(\Omega x)] \begin{pmatrix} \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} \\ 1 \end{pmatrix} + \begin{pmatrix} u_{1,p} \\ v_{1,p} \end{pmatrix}.$$

Using the condition $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in U^2 = \left(C_0^2[0,1]\right)^2$, we have

$$C_1 \begin{pmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{\lambda_0} \\ 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \lambda_0 + \mu_2 n d_0 \\ -\mu_2 n \end{pmatrix} = 0, \tag{4.13}$$

$$C_1 \begin{pmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} \\ 1 \end{pmatrix} + (C_3 cos\Omega + C_4 sin\Omega) \begin{pmatrix} \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{\lambda_0} \\ 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \lambda_0 + \mu_{2n} d_0 \\ -\mu_2 n \end{pmatrix} = 0, \tag{4.14}$$

thus $C_3(cos\Omega - 1) + C_4 sin\Omega = 0$, (4.12) implies $C_2 = 0$.

Therefore, the solution of (4.8) subject to the constraint condition (4.7) is

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = AC_1 cos(n\pi x) \begin{pmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} \\ 1 \end{pmatrix} + A[C_3 cos(\Omega x) + C_4 sin(\Omega x)] \begin{pmatrix} \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} \\ 1 \end{pmatrix}$$
$$+A \begin{pmatrix} -\frac{1}{\lambda_0} \\ 0 \end{pmatrix} + \frac{A}{k} \begin{pmatrix} \lambda_0 + \mu_{2n} d_0 \\ -\mu_2 n \end{pmatrix} cos(2n\pi x), \tag{4.15}$$

where (C_1, C_3, C_4) is the unique solution of the following linear algebraic system

$$\begin{pmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} & \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} & 0 \\ 1 & 1 & 0 \\ 0 & cos\Omega - 1 & sin\Omega \end{pmatrix} \begin{pmatrix} C_1 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_0} - \frac{1}{k}(\lambda_0 + \mu_2 n d_0) \\ \frac{1}{k}\mu_{2n} \\ 0 \end{pmatrix}.$$

In fact, the determinant

$$\begin{vmatrix} \frac{(q-2)\lambda_0}{\mu_n+\lambda_0} & \frac{(q-2)\lambda_0}{\Omega^2+\lambda_0} & 0 \\ 1 & 1 & 0 \\ 0 & cos\Omega - 1 & sin\Omega \end{vmatrix} = \frac{sin\Omega(q-2)\lambda_0(\Omega^2 - \mu_n)}{(\lambda_0 + \mu_n)(\lambda_0 + \Omega^2)} \neq 0.$$

Finally, taking the scalar product of (4.2) with $\Phi_n^*(\lambda_0, d_0)$ and using (4.15), after some straightforward but tedious calculations, we have

$$\lambda_2(\lambda_0) = \frac{-\lambda_0}{\langle \mathcal{L}_1 \Phi_n, \Phi_n^*(\lambda_0, d_0) \rangle} \left[N^{(2)} \left(\frac{(q-2)\lambda_0}{\mu_n + \lambda_0}, 1 \right) \left\langle v_1 \varphi_n \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \right.\right.$$
$$\left. \times \Phi_n^*(\lambda_0, d_0) \right\rangle + \left\langle [(2-q)u_1 + (1-q)v_1]\varphi_n \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \Phi_n^*(\lambda_0, d_0) \right\rangle$$
$$\left. + N^{(3)} \left(\frac{(q-2)\lambda_0}{\mu_n + \lambda_0}, 1 \right) \left\langle \varphi_n^3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \Phi_n^*(\lambda_0, d_0) \right\rangle \right]$$
$$= \frac{-\mu_n \lambda_0}{(\mu_n + \lambda_0) < \mathcal{L}_1 \Phi_n, \Phi_n^*(\lambda_0, d_0) >} \left\{ N^{(2)} \left(\frac{(q-2)\lambda_0}{\mu_n + \lambda_0}, 1 \right) \int_0^1 v_1 \varphi_n^2 dx \right.$$
$$\left. + \int_0^1 [(2-q)u_1 + (1-q)v_1]\varphi_n^2 dx + \frac{3}{8} N^{(3)} \left(\frac{(q-2)\lambda_0}{\mu_n + \lambda_0}, 1 \right) \right\}.$$

Then the bifurcation direction is $\lambda = \lambda_0 + \lambda_2(\lambda_0)s^2 + o(s^2)(s \to 0)$, the pitchfork bifurcations are subcritical or supercritical according to $\lambda_2(\lambda_0) < 0$ or $\lambda_2(\lambda_0) > 0$, respectively.

5 Conclusion

The mixed-order autocatalysis kinetic system (1.1) is worth considering [1]. Scott S.K. and Showalter K. [13] analysed the existence of constant speed traveling waves in such system. A real life example is the iodate-arsenite reaction, it was studied extensively in Saul A. and Showalter K [12].

The mathematical analysis carried out in this paper is focused on the local behavior of the curves bifurcating from the known state, the knowledge of the first non-vanishing term in asymptotic expansion of bifurcation parameter $\lambda = \lambda_0 + \lambda_2(\lambda_0)s^2 + o(s^2)$ is crucial in estimating the size of the continuation step.

In general, the calculation of λ_i is usually tedious, numerical analysis is needed, but the mathematical analysis can aid the numerical study of a real world problem. We have introduced the bifurcation parameter λ in the reaction diffusion system. This is the most common formulation. However, other parameters are also reasonable bifurcation parameters, for example, the ratio of diffusion coefficients δ.

To compare the decades old work by Auchmuty et al. [1], Herschkowitz-Kaufman [4], we have shown that the spatial structures formed by the parameter $p \in [0, 1]$ are richer than those formed by the purely quadratic step for $p = 0$ and the purely cubic step for $p = 1$ respectively (see the bifurcation directions in Sect. 4).

Acknowledgements We are grateful to the referees for their careful reading the manuscript and their many valuable comments and suggestions that greatly improved the presentation of this work. The research has been supported by the National Natural Science Foundation of China 10871078 and 30873121.

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